1. How much can this winch lift if 50 lbs of force are applied to the handle with a 15 inch sweep arm?

(A) 25 lbs.
(B) 50 lbs.
(C) 100 lbs.
(D) 250 lbs.
(E) 500 lbs.

By turning the crank, a winch converts angular force into linear force. The amount of work done is force times distance. The force moves the handle in a circle, and the rope in a linear direction.

The amount of work done by the handle has to be the same as that done by the rope. So force times distance at the handle has to equal the gravitational force exerted on the load, times the distance the rope moves.

With each complete turn of the handle, the handle moves a distance of $2\pi$ times a radius of 15 inches, while the rope moves a distance of $2\pi$ times a radius of 3 inches which is the circumference of the cylinder. Since force times distance is the same, the amount of work needed to move 50 pounds a distance of $2\pi$ times 15 inches equals the amount of work needed to move 250 pounds a distance of $2\pi$ times 3 inches.

\[
F \times 2\pi \times 3 \text{ in} = 50 \text{ lbs} \times 2\pi \times 15 \text{ in}
\]
\[
F = 250 \text{ lbs}
\]

2. A ball is steadily accelerated from rest in a circular direction with an angular acceleration of 4 radians per second squared ($4 \text{ rad sec}^{-2}$). How long will it take the ball to make three revolutions?

(A) $3\pi$ sec
(B) $3.5\pi$ sec
(C) $\sqrt{(3.5\pi)}$ sec
(D) $\sqrt{(3\pi)}$ sec
(E) $3.7\pi$ sec

In an angular velocity versus time graph, angular velocity is omega, in radians per second, along the Y axis.
Angular acceleration, \(\alpha\), is angular velocity, \(\omega\), over time, in radians per second squared. Angular acceleration is the slope of the graph line.

The distance traveled along the circumference is the area under the graph line. If the objects starts from zero velocity, the area under the graph line is one-half the angular velocity, \(\omega\), times time, in other words, the area of the gray triangle.

Angular velocity, \(\omega\), over time, is angular acceleration, \(\alpha\). A complete revolution of 360 degrees is \(2\pi\) radians.

In this problem, angular acceleration, \(\alpha\), or \(\omega / t\), is 4 radians per second squared \((\alpha = \omega / t - 4 \frac{r}{sec^2})\).

Omega equals 4 radians per second squared times \(t\) \((\omega = 4 \frac{r}{sec^2} \times t)\).

The distance traveled is three complete revolutions, or \(6\pi\). Since the distance of \(6\pi\) equals one-half of angular velocity, \(\omega\), times \(t\), \(6\pi\) equals one-half times 4 radians per second squared times \(t\) squared.

\[
6\pi = \frac{1}{2} \omega \times t
\]

\[
6\pi = \frac{1}{2} \left(4 \frac{r}{sec^2}\right) \times t \times t
\]

\[3\pi (sec^2) = t^2\]

\(t\) works out to be \(\sqrt{3\pi} \text{ sec}\).

\[t = \sqrt{3\pi} \text{ sec}\]

In the angular velocity versus time graph, the area under the graph line is the distance traveled, which, for 2 revolutions, is now \(4\pi\). The final angular velocity is unknown, so we’ll call it \(X \frac{rad}{sec}\).

The area under the graph line is a rhomboid, so its area is \(\frac{X+2}{2 \frac{rad}{sec}} \times t\), the time it takes to go 2 revolutions, a distance of \(4\pi\). What is \(t\) seconds?

We know that the ball accelerates at \(4 \frac{rad}{sec^2}\).
Angular acceleration, alpha, is the increase in angular velocity, omega, divided by the time it took to make that increase in angular velocity.

In this case, the increase in angular velocity is $X - 2 \frac{rad}{sec}$.

\[
\frac{X - 2 \frac{rad}{sec}}{t}, \text{ the time it took increase the angular velocity, equals the angular acceleration of } 4 \frac{rad}{sec^2}.
\]

\[t = \frac{(X - 2) \frac{rad}{sec}}{4 \frac{rad}{sec^2}} \text{ or } (X - 2) \frac{sec}{4} \]

Plugging this value of $t$ into the distance equation, we get:

\[4\pi \text{ equals } (X + 2) \frac{rad}{sec} \text{ divided by } 2, \text{ times } (X - 2) \frac{sec}{4}.\]

\[4\pi = \frac{(x + 2) \frac{rad}{sec}}{2} \times \frac{(x - 2) \frac{sec}{4}}{4}\]

$X$ works out to be two times the square root of $8\pi + 1$ radians per second.

\[x = 2\sqrt{8\pi + 1} \frac{rad}{sec}\]

Since angular velocity, omega, times the radius is tangential velocity, v, the tangential velocity is 2 times the square root of $8\pi + 1$ radians/second times 20 centimeters, or 40 times the square root of $8\pi + 1$ centimeters/sec.

\[(2\sqrt{8\pi + 1} \frac{rad}{sec})(20 \text{ cm}) = 40\sqrt{8\pi + 1} \frac{cm}{sec}\]

4. A ball weighing 4.9 newtons is swung in a circle at $5 \frac{m}{sec}$ using a 5 meter wire. What force is exerted by the wire?

(A) 2.5 N
(B) 0.5 N
(C) 25.0 N
(D) 9.8 N
(E) 4.9 N

The wire exerts a centripetal force. Centripetal force is $\frac{mv^2}{r}$. 
The weight of something is its mass times the acceleration of gravity.

\[
\text{centripetal force} = \frac{mv^2}{r}
\]

weight = \( m \times g \)

\[
4.9 \text{ N} = 4.9 \text{ kg-m/sec}^2 = m \times 9.8 \text{ m/sec}^2
\]

\[
m = 0.5 \text{ kg}
\]

\[
\text{centripetal force} = (0.5 \text{ kg})(5.0 \text{ m/sec})^2/5.0 \text{ m}
\]

\[
\text{centripetal force} = 2.5 \text{ kg-m/sec}^2 = 2.5 \text{ N}
\]

5. The next five questions refer to a ball at rest that is suddenly spun horizontally by a 3 meter string with an angular acceleration of 2\(\pi\) radians per second squared. After 5 seconds, what is its angular velocity?

(A) \(2\pi \frac{\text{rad}}{\text{sec}}\)

(B) \(4\pi \frac{\text{rad}}{\text{sec}}\)

(C) \(6\pi \frac{\text{rad}}{\text{sec}}\)

(D) \(8\pi \frac{\text{rad}}{\text{sec}}\)

(E) \(10\pi \frac{\text{rad}}{\text{sec}}\)

What do we know about circular motion? We know that omega is angular velocity in radians per second.

Angular velocity over time is angular acceleration, alpha, which means that angular velocity is angular acceleration times time.

1 revolution is 360 degrees or 2\(\pi\) radians. Tangential velocity is the radius times angular velocity, and conversely, angular velocity is tangential velocity divided by the radius.

Likewise, tangential acceleration is the radius times angular acceleration, and angular acceleration is tangential acceleration divided by the radius.

We know that centripetal acceleration is tangential velocity squared divided by the radius.

In this case, angular acceleration is 2\(\pi\) radians per second squared (\(\alpha = \frac{\omega}{t} = 2\pi \text{ rad/sec}^2\)).

Since angular acceleration times time is angular velocity, 2\(\pi\) radians per second squared times 5 seconds is 10\(\pi\) radians per second (\(\omega = 2\pi \text{ rad/sec}^2 (5 \text{ sec})\)).

In English, angular velocity increases by 2\(\pi\) every second, so after 5 seconds the angular velocity is 2\(\pi\) times 5 seconds, or 10\(\pi\) radians per second.
6. How many revolutions did the ball make after 5 seconds?

(A) 5 revolutions
(B) 7.5 revolutions
(C) 10 revolutions
(D) **12.5 revolutions**
(E) 15 revolutions

Distance traveled is the area under the graph line. For an object starting out at rest, the area under the graph line is \( \frac{1}{2} \) the final angular velocity times time

\[
\frac{1}{2} \times 10 \, \text{rad/sec} \times 5 \, \text{seconds} = 25 \pi \, \text{radians}
\]

Since 1 revolution is \( 2\pi \), \( 25\pi \) must be 12.5 revolutions \( (25\pi \, \text{rad} = 12.5 \, \text{rev}) \).

7. What is its tangential velocity after 5 seconds?

(A) \( 25\pi \, \text{m/sec} \)
(B) \( 30\pi \, \text{m/sec} \)
(C) \( 35\pi \, \text{m/sec} \)
(D) \( 40\pi \, \text{m/sec} \)
(E) \( 45\pi \, \text{m/sec} \)

Tangential velocity is the radius of rotation times angular velocity \( (v = r \omega) \).

In question 6 we determined that angular velocity after 5 seconds was \( 10\pi \) radians per second.

If tangential velocity is the radius of rotation times angular velocity, then tangential velocity is 3 meters times \( 10\pi \, \text{rad/sec} \), or \( 30\pi \, \text{m/sec} \).

\[
v = 3 \, \text{m} \times (10\pi \, \text{rad/sec})
\]

\[
v = 30\pi \, \text{m/sec}
\]
8. What is its centripetal acceleration after 5 seconds?
(A) $150\pi \frac{m}{sec^2}$
(B) $200\pi \frac{m}{sec^2}$
(C) $250\pi \frac{m}{sec^2}$
(D) $300\pi \frac{m}{sec^2}$
(E) $350\pi \frac{m}{sec^2}$

Centripetal acceleration is $v^2/r$.

$30 \pi \frac{m}{sec^2} = \frac{900 \pi \frac{m^2}{sec^2}}{3 \text{ meters}^2}$, or $\frac{m}{sec^2}$

9. What is its tangential acceleration after 5 seconds?
(A) $2\pi \frac{m}{sec^2}$
(B) $4\pi \frac{m}{sec^2}$
(C) $6\pi \frac{m}{sec^2}$
(D) $8\pi \frac{m}{sec^2}$
(E) $10\pi \frac{m}{sec^2}$

Tangential acceleration is the radius of rotation times angular acceleration, $\alpha$.

Angular acceleration was given as $2 \pi \frac{rad}{sec^2}$.

$2 \pi \frac{rad}{sec^2} \times 3 \text{ meters} = 6 \pi \frac{m}{sec^2}$.

$A_{\text{tangential}} = (3 \text{ m}) (2 \pi \frac{rad}{sec^2}) = 6 \pi \frac{m}{sec^2}$

Neither the angular nor the tangential acceleration changes during the period of rotation.

10. This ball with a mass of 1 kg is being swung in a vertical circle by a 4 meter wire. The force exerted by the wire is 10.2 newtons. What is the ball’s tangential velocity when the ball is at 12 o’clock high?
Tangential velocity is symbolized by the letter $v$.

Tangential velocity, $v$, can be calculated from the angular velocity, omega, because the radius times angular velocity is tangential velocity.

Tangential velocity can also be calculated from the centripetal acceleration, because centripetal acceleration equals $v$ squared over $r$.

Centripetal acceleration times the mass of the ball is the centripetal force pulling the ball toward the center of rotation.

When the ball is at 12 o’clock high, there are two forces making up the centripetal force. One is the 10.2 newtons of force exerted by the wire, and the other is the force of gravity, which is the mass of the ball, 1 Kg, times 9.8 $\frac{m}{sec^2}$, or 9.8 newtons.

$$F_{\text{centripetal}} = 10.2N + 9.8N = 20N$$

The total centripetal force exerted on the ball at 12 o’clock is 10.2 plus 9.8, or 20 newtons. 20.0 newtons is the centripetal force pulling the ball toward the center of rotation.

$$F_{\text{centripetal}} = 20.0N = m \frac{v^2}{r}$$

Centripetal force is the mass of the ball times its centripetal acceleration, or the mass of the ball times $v$ squared over $R$.

11. At what tangential velocity must a 200 gram ball be spun in the vertical direction by a 5 meter string in order for the ball to maintain a circular path?

(A) 4.0 $\frac{m}{sec}$
(B) 5.0 $\frac{m}{sec}$
(C) 6.0 $\frac{m}{sec}$
(D) 7.0 $\frac{m}{sec}$
(E) 8.0 $\frac{m}{sec}$

When you spin a ball horizontally on a string, the velocity of the ball depends on the centripetal force you exert on the string. The centripetal force is provided by the string tension, T.

The more force you exert on the string, the faster the ball spins around, because, mathematically, the centripetal force you exert on the string has to equal $mv$ squared over $r$. 

$$20.0 Kg \frac{m}{sec^2} \text{ equals } 1 \text{ Kg times } v^2/4 \text{ m}$$

$$20.0 kg \frac{m}{sec^2} = 1 \text{ kg } \frac{v^2}{4 \text{ m}}$$

$$v^2 = 80.0 \frac{m^2}{sec^2} = 4 \times \sqrt{5 \frac{m}{sec}}.$$
Since you’re holding the string at one end, the string cannot shorten and R remains constant. The only way for \( \frac{mv^2}{r} \) to increase when the centripetal force is increased is to increase v, the tangential velocity of the ball.

When a ball is swung vertically, the centripetal force is made up of the tension force you exert on the string, plus whatever effect the force of gravity has in that same direction.

For example, when the ball is at 12 o’clock high, the weight of the ball, mg, is directed in the same direction as the string, so the centripetal force is \( T + mg \).

So long as these two forces, combined, are less than \( \frac{mv^2}{r} \), the ball rotates in a circle.

When the ball rotates over to the 3 o’clock and 9 o’clock positions, the force of gravity is perpendicular to the direction of the string, and therefore makes no contribution to the centripetal force, which is directed along the string toward the center of rotation.

At the bottom of the circular path, the force of gravity opposes the tension in the string, so the centripetal force at the 6 o’clock position is \( T - mg \), and the centripetal force of \( \frac{mv^2}{r} \) now equals \( T - mg \).

What this formula says is that at the bottom of the rotation, the tension in the string, \( T \), equals \( \frac{mv^2}{r} + mg \). Think about when you spin a ball on a string. Don’t you recall feeling this increase in string tension at the bottom of the rotation?

At the top of the rotation, where \( T = \frac{mv^2}{r} - mg \), you feel less tension in the string.

What \( T + mg \) equals \( \frac{mv^2}{r} \) is saying mathematically about the ball at the top of the rotation is that so long as \( T = \frac{mv^2}{r} - mg \) is greater than zero, the ball will revolve in a circle.

So long as the tension in the string does not drop below zero, the ball will revolve in a circle. If the force of gravity, mg, does become greater than the centripetal force, the ball will fall when it reaches 12 o’clock high.

String tension drops below zero when the acceleration of gravity becomes greater than the ball’s centripetal acceleration, \( \frac{v^2}{r} \).

So long as centripetal acceleration, \( \frac{v^2}{r} \), is equal to or greater than the acceleration of gravity, meaning so long as tangential velocity is equal to or greater than the square root of \( r \) times \( g \), the ball will remain in a circular path.
In this problem, $v$ equals the square root of $5.0$ meters times $9.8 \, \frac{m}{sec^2}$, or the square root of $49 \, \frac{m^2}{sec^2}$.

$$v = \sqrt{5 \times 9.8 \, \frac{m}{sec^2}} = \sqrt{49 \, \frac{m^2}{sec^2}} \, \text{change}$$

$v = 7 \, \frac{m}{sec}$

So long as the ball maintains a tangential velocity equal to or greater than $7 \, \frac{m}{sec}$, the ball will continue to spin in a circle. Notice that the mass of the ball is unimportant.

12. What is the tension in this 5 meter string when the 200 gram ball being spun vertically at $7 \, \frac{m}{sec}$ reaches the 4 o’clock position in its rotation?

(A) 2.9 N  
(B) 3.4 N  
(C) 3.9 N  
(D) 4.4 N  
(E) 4.9 N

We know from the previous problem that the centripetal force pulling the ball toward the center of rotation is made up of two forces: the tension in the string, symbolized by vector $T$, plus the force of gravity. These two forces, the tension in the string and the force of gravity, together, form the centripetal force equaling $\frac{mv^2}{r}$.

The gravitational force on the ball is aimed downward, but a component of that force lies along the centripetal force vector, pointing away from the center of rotation.

That component is $mg$ times the cosine of theta.

$$\text{centripetal force} = \frac{mv^2}{r}$$

$$\frac{mv^2}{r} = T - mg \cos \theta$$

Since the gravitational component opposes the tension in the string, the centripetal force is made up of $T$ pointing inward minus $mg$ times the cosine of theta pointing outward.
When the ball reaches the 4 o’clock position, theta is 60 degrees, and the cosine of 60 degrees is 0.5.

The centripetal force, $mv^2/r$, therefore equals $T - mg \times \frac{1}{2}$.

Centripetal force $= \frac{mv^2}{r} = T - mg \times \frac{1}{2}$

When the mass of 0.2 kilograms, and the velocity of $7 \frac{m}{sec}$, and the cosine of 60 degrees are inserted into the equation, the tension in the string works out to be 2.9 newtons.

$(0.2 \text{kg})(7.0 \frac{m}{sec})^2/5.0 \text{ m} = T - (0.2 \text{ kg})(9.8 \frac{m}{sec^2})(0.5)$

$T = 2.9 \text{ kg-m/sec}^2 \text{ (N)}$

Notice that when the ball reaches 6 o’clock, theta becomes zero degrees, and the cosine of theta becomes 1. The tension in the string, symbolized by the Letter T, becomes mv squared over r plus mg $(\frac{v^2}{r + mg})$, which is what we found in the last problem.

The centripetal acceleration of an object moving in a circular path is its velocity squared divided by the radius of rotation.

$\frac{(313 \frac{m}{sec})^2}{1000 \text{ m}} = \frac{97,969}{1000} = 98.0 \frac{m}{sec^2}$

$\frac{98.0 \frac{m}{sec^2}}{9.8 \frac{m}{sec^2}} = 10.0 \text{g}$

13. Fighter jets often have to perform steep dives. How many g’s of acceleration must the pilot of this jet airplane endure if he enters the dive at 313 meters per second and has to pull out of the dive along a circular path of 1000 meter radius?
At the beginning of its climb out of the dive, the plane and the pilot have to withstand 10 times the acceleration of gravity.

As we learned from twirling a ball on a string in the vertical direction, at the bottom of the circular path, \( v^2 / r \) is not the only force being exerted on the plane. Gravity is also accelerating the plane downward, so the pilot and the plane have to withstand not only centripetal acceleration but also the additional downward acceleration of gravity.

To come out of the dive, then, the pilot and his plane have to withstand 11 g’s of acceleration.

14. A 13.0 kg pulley with a radius of 2.0 meters is being spun by a falling 1.0 kg block. What is the angular acceleration and the tangential acceleration of the pulley?

\[
a = \frac{v^2}{r} = \frac{(313 \text{ m/sec})^2}{1000 \text{ m}} = 98.0 \text{ m/sec}^2 = 10.0 \text{ g}
\]

If we were to simply drop an unattached 1 kg mass, it would fall much faster than this 1 kg mass attached to the pulley. The upward tension in the wire prevents the 1 kg mass from free-falling.

To express this mathematically, the net downward force, \( F \), acting on the 1 kg mass equals the force of gravity, \( m \times g \), minus the upward tension of the wire, \( T \).

\[
F_{\text{net}} = mg - T
\]

Since force is mass times acceleration, \( mg \) minus \( T \) is causing the 1 kg mass to accelerate downward with an acceleration of \( a \).

\[
Ma = mg - T
\]
We need to find the value of $T$, the tension in the wire, because that upward force is reducing the weight of the 1 kilogram mass by an amount equal to $T$.

$$T = mg - ma$$

The tension in the wire is exerting a force on the circumference of the pulley. Because the force is some distance from the axis of rotation, the tension in the wire is exerting a torque on the pulley and causing its moment of inertia to spin with an angular acceleration of alpha.

$$\tau = I \alpha$$

The torque exerted by the tension in the wire is $T$ times the radius, $r$.

The moment of inertia for a disc is one-half its mass times its radius squared.

So, $T$ times $r$ equals $\frac{1}{2} M r^2 \alpha$.

Solving for $T$, we get $T$ equal to one-half the mass of the pulley, capital M, times its radius, $r$, times its angular acceleration, alpha.

$$T = \frac{1}{2} M r \alpha$$

The radius of a disc times its angular acceleration, $r$ times alpha, is its tangential acceleration, $a$, so $T$ equals one-half the mass of the pulley times its tangential acceleration, $a$.

Because the wire is attached to both the pulley and the 1 kilogram block, the tension spinning the pulley is the same tension resisting the block’s descent.

Therefore, the net force pulling the 1 kg mass downward, $mg$ minus $ma$, equals the tangential force rotating the pulley, one-half the mass of the pulley times its tangential acceleration.

$$mg - ma = \frac{1}{2} Ma$$

This means that the weight of the 1 kilogram mass, $mg$, has been reduced by one-half the mass of the pulley times its tangential acceleration.

$$Ma = mg - \frac{1}{2} Ma$$

Solving for $a$, which represents both the downward acceleration of the 1 kg mass and the tangential acceleration of the pulley, and then substituting actual numbers into the equation, we get a downward acceleration of $2.8 \frac{m}{sec^2}$.

$$a = \frac{2mg}{2m+M}$$

$$a = \frac{2(1.0\,\text{kg})(9.8\,\frac{m}{sec^2})}{(7.0\,\text{kg})}$$

$$a = 2.8 \frac{m}{sec^2}$$
Since the tension in the wire equals one-half the mass of the pulley times its angular acceleration, the tension in the wire is one-half of 5 kilograms times 2.8 meters per second squared, or 7 kg-meters per second squared, which is 7 newtons.

\[ T = \frac{1}{2} (5.0 \text{ Kg}) (2.8 \frac{m}{\text{sec}^2}) \]

\[ T = 7.0 \text{ kg} \cdot \frac{m}{\text{sec}^2} (N) \]

In the next lesson, we will see other examples of the weight of a mass being reduced by some upward force, either the tension on a string or the upward force of buoyancy when the mass is lowered into a jar of water.

15. What is the angular velocity of the pulley after it has made 3 revolutions?

(A) \(2\sqrt{21\pi}\) \(\frac{\text{rads}}{\text{sec}}\)

(B) \(4\sqrt{21\pi}\) \(\frac{\text{rads}}{\text{sec}}\)

(C) \(4\sqrt{41\pi}\) \(\frac{\text{rads}}{\text{sec}}\)

(D) \(2\sqrt{41\pi}\) \(\frac{\text{rads}}{\text{sec}}\)

(E) \(6\sqrt{41\pi}\) \(\frac{\text{rads}}{\text{sec}}\)

Every mass resists being accelerated or decelerated. This resistance is called inertia. For linear motion, inertia depends only on the number of kilograms in the mass. For rotational movements, however, inertia also depends on how far away the mass is located from the axis of rotation. This inertia is called rotational inertia, or moment of inertia.

The distance of a rotating mass from its axis of rotation is called its moment arm. The longer the moment arm, the greater the rotational inertia of the rotating mass. You have probably experienced this when spinning a wheel. It takes more effort to get a wheel spinning when the mass is concentrated in the rim than when the mass is concentrated around the axle.

The force overcoming rotational inertia, or the moment of inertia, is called torque.

Rotational inertia is symbolized by the capital letter \(I\).

Rotational, or angular, acceleration is symbolized by the Greek letter alpha (\(\alpha\)), measured in radians per second squared, and torque is symbolized by the Greek capital letter \(T\) (\(\tau\) or Tau).
The chart on the next slide summarizes the relation between linear, angular, tangential, and centripetal movements. (next slide)

Torque has two mathematical definitions, one for the moment arm and another for the location of the mass around the axis of rotation. Torque is force times the radius of rotation, $F \times r$. The other definition of torque is rotational inertia times angular acceleration, $I \times \alpha$. Angular velocity is $\omega$ and its units are rads per second $\left(\frac{\text{rads}}{\text{sec}}\right)$.

The work needed to get the pulley to rotate three revolutions is force times distance. That work energy is converted to a kinetic energy equal to $\frac{1}{2} I \omega^2$.

Expressed mathematically, force times distance equals one-half $I \omega^2$.

$Fd = \frac{1}{2} \omega^2$

Force times distance is the force applied to the circumference of the pulley times the distance the circumference moves. In the previous problem, we determined that the falling mass exerted a tension in the wire of $7 \text{kg} \cdot \frac{\text{m}}{\text{sec}^2}$. The tension was applied to the circumference of the pulley.

In three revolutions, the circumference of the pulley moves a distance of $3 \times 2\pi r$, or $6 \pi$ times the radius. So the work of rotating the pulley 3 revolutions is $7 \text{kg} \cdot \frac{m}{\text{sec}^2} \times 6 \pi \times$ the radius of 0.1 meter, or $4.2 \pi \text{kg} \cdot \frac{m^2}{\text{sec}^2}$.

This force times distance equals the kinetic energy of the rotating pulley, $\frac{1}{2} I \omega^2$.

$Fd = (7.0 \text{ kg} \cdot \frac{m}{\text{sec}^2})(6\pi)(0.1\text{ m})$

$Fd = 4.2 \pi \text{ kg} \cdot \frac{m^2}{\text{sec}^2}$

The pulley’s moment of inertia, $I$, is $\frac{1}{2} mr^2$. So, $4.2 \pi \text{ kg} \cdot \frac{m^2}{\text{sec}^2}$ equals $\frac{1}{2}$ times $\frac{1}{2} m r^2$ times $\omega^2$.

$4.2 \pi \text{ kg} \cdot \frac{m^2}{\text{sec}^2} = \frac{1}{2} \left(\frac{1}{2} mr^2\right) \omega^2$

Omega works out to be $4\sqrt{21}\pi$.

$\omega^2 = 16 \times 21\pi \frac{\text{rads}^2}{\text{sec}^2}$

$\omega = 4 \sqrt{21}\pi \frac{\text{rads}}{\text{sec}}$

$\frac{\text{F d}}{\text{L}} = \frac{1}{2} \left(\frac{1}{2} \frac{m}{\text{sec}^2}\right) \omega^2$

$F d = 4.2\pi \text{ kg} \cdot \frac{m^2}{\text{sec}^2} = \frac{1}{2} \left(5.0 \text{ kg} \cdot (0.1 \text{ m})^2\right) \omega^2$

$F d = (7.0 \text{ kg} \cdot \frac{m}{\text{sec}^2})(6\pi)(0.1\text{ m})$

$4.2\pi \text{ kg} \cdot \frac{m^2}{\text{sec}^2} \cdot \frac{\omega^2}{\text{sec}^2} = 356.7\pi \text{ rad/sec}^2 \cdot \frac{\omega^2}{\text{sec}^2}$

$F d = 4.2\pi \text{ kg} \cdot \frac{m^2}{\text{sec}^2}$

$\omega = 4 \sqrt{21}\pi \frac{\text{rads}}{\text{sec}}$
16. When this 4.0 meter rod keels over, what is the tangential speed of the top of the rod just before impact?

(A) $6.2 \text{ m sec}^{-1}$ 
(B) $6.2 \text{ rad sec}^{-1}$ 
(C) $5.4 \text{ m sec}^{-1}$ 
(D) $10.8 \text{ m sec}^{-1}$ 
(E) $12.1 \text{ m sec}^{-1}$

At rest, all of the rod’s energy is potential energy. When the rod keels over, its potential energy is converted to an equivalent amount of kinetic energy.

So we begin with potential energy. What is the rod’s potential energy?

Potential energy is mass times the acceleration of gravity times the height above ground level.

$$PE = mg$$

In this case, mass times g is clear, but what is the rod’s height above the ground?

If all the mass of the rod were concentrated into one point, where would that point be? In other words, where is the rod’s center of gravity?

The center of gravity is halfway up the rod, because that’s where all the mass would be balanced.

So the potential energy of this vertical rod is its mass times the acceleration of gravity times half its length.

Potential energy equals mg times L/2.

$$PE = mg \frac{L}{2}$$

When the rod topples over, it rotates, so all of the potential energy is converted into rotational kinetic energy. Kinetic energy is $\frac{1}{2} m v^2$, but for rotational movement, kinetic energy is $\frac{1}{2} I \omega^2$.

$$KE = \frac{1}{2} I \omega^2$$

The moment of inertia for a rod is $\frac{1}{3} m L^2$. 

$$I = \frac{1}{3} m L^2$$
Omega is the angular velocity in rads per second. To convert it to meters per second, multiply omega by the radius. What is the radius of rotation for the top of the rod?

\[ \omega = \frac{v}{L} \]

L. Angular velocity omega, then, is tangential velocity divided by the length of the rod.

\[ mg \frac{L}{2} = \frac{1}{2} \left( \frac{1}{3} m L^2 \right) \frac{v^2}{L} \]

Since potential energy equals kinetic energy, \( mg \) times half the length of the rod equals one-half the moment of inertia of the rod times its angular acceleration, \( V \) over \( L \), squared.

The tangential velocity, \( v \), works out to be the square root of \( 3 g L \).

\[ 3 g L = v^2 \]

\[ v = \sqrt{3 g L} \]

When the values for \( g \) and \( L \) are inserted, the tangential velocity of the rod at impact is 10.8 m/sec.

\[ v = \sqrt{\left(3 \times 9.8\right) \frac{m}{sec^2} \times 4.0m} \]

\[ v = 10.8 \frac{m}{sec} \]

17. A 2 kg disc, 20 cm in diameter, spinning at 60 rads per second is brought to a halt with a tangential force of 2 newtons. How long does it take for the disc to come to a halt?

(A) 2.4 sec  
(B) 2.6 sec  
(C) 2.8 sec  
(D) 3.0 sec  
(E) 3.2 sec

The 2 newton tangential force exerted on the circumference of the disc exerts a torque equal to 2 newtons times the radius of rotation, 0.1 m. The torque is 0.2 newton meters, or 0.2 kg-meters squared per second squared, or 0.2 joules of energy.
This torque slows the angular velocity from 60 radians per second to zero. The rate of deceleration is 60 radians per second divided by the time, \( t \), that it takes for the disc to come to a complete halt.

\[
\alpha = \frac{60 \text{ rads/sec}}{t}.
\]

What is the relation between the torque exerted on a disc and the angular deceleration caused by that torque?

Torque equals the disc’s moment of inertia, \( I \), times its angular acceleration, \( \alpha \), \( (\tau = I \times \alpha) \). which, in this case, will be a negative number because the disc is decelerating.

We now have two ways to express \( \alpha \) (\( \alpha \)). One is omega over \( t \) \( (\frac{\omega}{t}) \), and the other is torque divided by the moment of inertia \( (\frac{T}{I}) \).

Since omega over \( t \) \( (\frac{\omega}{t}) \) and torque \( (\tau) \) divided by moment of inertia \( (\frac{\tau}{I}) \) both equal \( \alpha \), they equal each other.

\( t \), the time it takes for the disc to stop spinning, equals the angular velocity, omega, times the moment of inertia, \( I \), divided by the torque \( (t = \frac{\omega \times I}{\tau}) \).

The moment of inertia \( (I) \) for a disc is one-half its mass times its radius squared.

\[
I = \frac{1}{2}mr^2.
\]

Plugging in the values for omega, radius, and torque, time works out to be 3 seconds.

\[
t = \frac{(60 \text{ rad/sec})(0.1)(2.0 \text{ kg})(0.1)^2}{0.2 \text{ kg} \cdot \text{m}^2/\text{sec}^2}
\]

18. We’ve stopped a spinning disc. How about we try to stop the earth from spinning! How much force would it take to stop the earth within 1 hour?

(A) \( 3.1 \times 10^{23} \text{ N} \)
(B) \( 8.1 \times 10^{23} \text{ N} \)
(C) \( 3.1 \times 10^{21} \text{ N} \)
(D) \( 8.1 \times 10^{21} \text{ N} \)
(E) \( 3.1 \times 10^{25} \text{ N} \)
First, what is the angular velocity of the earth in rads?

The earth rotates once every 24 hours, or once every 86,400 seconds.

How many rads is one rotation?

$2\pi$. So the earth rotates at an angular velocity of $2\pi$ per 86,400 seconds ($\omega = \frac{2\pi \text{ rads}}{86,400 \text{ sec}}$), or 7.3 times 10 to the minus 5 rads per second $(7.3 \times 10^{-5} \text{ rads/sec})$.

In a graph of angular velocity versus time graph, we want to stop the earth’s rotation from 7.3 times 10 to the minus 5 rads per second $(7.3 \times 10^{-5} \text{ rads/sec})$, to zero over 1 hour.

1 hour contains 3600 seconds.

The rate of deceleration, alpha, is 7.3 times 10 to the minus 5 rads per second divided by 3600 seconds, or 2.0 times 10 to the minus 8 rads per second squared $(\alpha = \frac{7.3 \times 10^{-5} \text{ rads/sec}}{3600})$ or $(2.0 \times 10^{-8} \text{ rads/sec}^2)$.

Because we’re trying to stop a rotating body, we will need a torque to do it.

Torque, you recall from the chart a few slides ago, is both the moment of inertia, I, times angular acceleration, alpha, $(\tau = I \times \alpha)$ and force times the radius of rotation.

What is the earth’s moment of inertia? The moment of inertia for a solid sphere is two-fifths times its mass times its radius squared $(\frac{2}{5} mr^2)$.

The earth’s mass is 6 times 10 to the 24 kilograms and its radius is 6.4 times 10 to the 6 meters. Its moment of inertia is therefore two-fifths times 6 times 10 to the 24 kilograms times the square of 6.4 times 10 to the 6 meters $(I = \frac{2}{5} (6 \times 10^6 \text{ m})^2)$. The moment of inertia for the earth works out to 9.8 times 10 to the 37 kilogram-meters squared $(9.8 \times 10^{37} \text{ kgm}^2)$.

The torque needed to stop the earth’s rotation over 1 hour is the earth’s moment of inertia times its deceleration, I times alpha. That’s 9.8 times 10 to the 37 kilogram-meters squared times 2.0 times 10 to the minus 8 radians per second squared

$\tau = (9.8 \times 10^{37} \text{ kgm}^2 \times 2.0 \times 10^{-8} \text{ rads/sec}^2) \times \text{change}$

The torque needed is 19.6 kg times 10 to the 29 meters squared per second squared, or 19.6 times 10 to the 29 newton meters.

$\tau = 19.6 \times 10^{29} \text{ kg} \times \frac{m^2}{\text{sec}^2} (N - m)$

Torque is not force. The question asked for the force needed to stop the earth from rotating. What’s the relation between torque and force? Torque is force times the radius to the axis of rotation, which in this case is the radius of the earth.
\[ \tau = F_r \]

The force exerting the torque is tangential force. 19.6 times 10 to the 29 newton meters divided by 6.4 times 10 to the 6 meters is 3.1 times 10 to the 23 newtons of force.

\[
F = \frac{(19.6 \times 10^{29} \text{ kg} \cdot \text{m}^2/\text{sec}^2)}{6.4 \times 10^6 \text{ kg} \cdot \text{m}/\text{sec}^2} \\
\]

\[ F = 3.1 \times 10^{23} \text{ kg} - \frac{m}{\text{sec}^2} \cdot (N) \]

19. How does the force needed to stop the world from rotating over the course of 1 hour compare to the force exerted by the sun on the earth?

(A) 2.5 times as much
(B) 6.6 times as much
(C) 8.6 times as much
(D) 10.5 times as much
(E) 12.7 times as much

The sun exerts a force on the earth through gravity. The formula for the gravitational pull of the sun on the earth, and the earth on the sun, is \( F = G \frac{m_1 m_2}{r^2} \).

\[
F = G \frac{m_1 m_2}{r^2} \quad G = 6.7 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2 \\
\]

The mass of the sun is 2 times 10 to the 30 kilograms, the mass of the earth 6 times 10 to the 24 kg, and the distance between the sun and the earth is 1.5 \( \times \) 10\(^{11}\) meters.

The gravitational force between them works out to be 3.6 times 10\(^{22}\) newtons

The tangential force of 3.1 \( \times \) 10\(^{23}\) N needed to stop the earth’s rotation over 1 hour is 8.6 times greater than the sun’s gravitational force on the earth. This means that if the force of gravity exerted by the sun were used to stop the earth’s rotation, it would take 8.6 hours to stop the earth.

\[
F = \frac{(6.7 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2) (2.0 \times 10^{30} \text{ kg}) (6.0 \times 10^{24} \text{ kg})}{(1.5 \times 10^{11} \text{ m})^2} \\
F_{\text{gravitational}} = 3.6 \times 10^{22} \text{ N} \\
F_{\text{tangential}} - 3.1 \times 10^{23} \text{ N} \\
\frac{3.1 \times 10^{23} \text{ N}}{3.6 \times 10^{22} \text{ N}} = 8.6
\]
20. How much more work energy is expended by a skater spinning at 1.5 revolutions per second when she folds her arms in and reduces her moment of inertia from 4.5 kg·m² to 1.5 kg·m²?

With the arms folded in, the kinetic energy is one-half 1.5 kg·m² times her angular velocity squared.

\[ \text{KE} = \frac{1}{2} (1.5 \text{ kg} \cdot \text{m}^2) (?? \text{ rev/sec})^2 \]

What we don’t know is her angular velocity with her arms folded in.

Spinning objects have a momentum just like objects moving in a straight line, and unless a force acts on a spinning object, it continues spinning in the same direction. Angular momentum is conserved.

The angular momentum of a spinning object is its moment of inertia, I, times its angular velocity, omega. Unless a force acts on it, the angular momentum, I omega, remains the same.

The angular momentum of the skater before folding her arms in is 4.5 kg·m² times an angular velocity of 1.5 revolutions per second. The angular momentum after folding her arms in is now 1.5 kg·m² times a new and faster angular velocity. Since the total angular momentum doesn’t change, 4.5 kg·m² times 1.5 revolutions per second must equal 1.5 kg·m² times the new angular velocity.

\[ \text{KE} = \frac{1}{2} (4.5 \text{ kg} \cdot \text{m}^2) (1.5 \text{ rev/sec})^2 \]

The new angular velocity, omega, is 4.5 rev/sec.
Since 1 revolution is $2\pi$ radians, the kinetic energy before her arms were folded in is one-half times $4.5 \text{ kg-m}^2$ times $1.5$ times $2\pi$ radians/sec, squared, or 199.7 N-m, or 199.7 joules.

$$KE=199.7 \text{ kg-m}^2\text{sec}^{-2}(\text{N-m})$$

The kinetic energy after the arms are folded in is one-half $1.5 \text{ kg-m}^2$ times $4.5$ times $2\pi$ radians per second, squared, or 599.0 joules. When folding her arms in, the skater has to exert three times as much energy to spin three times faster. That makes sense.

$$KE=599.0 \text{ kg-m}^2\text{sec}^{-2}(\text{N-m})$$

$$\frac{599.0 \text{ N-m}}{199.7 \text{ N-m}} = 3$$

The second observation explained by the conservation of angular momentum is that satellites and comets speed up as they approach the earth if their orbits are not perfectly circular.

Angular momentum, $L$, is an object’s moment of inertia, $I$, times its angular velocity, $\omega$.

$$L=I\omega$$

The moment of inertia for a small mass rotating in a circular path is its mass times the radius of rotation squared.

$$I=mr^2$$

Its angular velocity, $\omega$, is its tangential velocity, $v$, divided by the radius of rotation.

$$\omega = \frac{v}{r}$$

So the angular momentum, $I\omega$, for a satellite is its mass times its radius of rotation times its velocity.

$$L=mrv$$

Because the satellite or comet revolves around the earth, the radius of rotation changes throughout its orbit. At the apogee, the radius is capital $R$, while at the perigee, the radius is small $r$. 

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A constant angular momentum, however, means, \( m r v \), is same everywhere in the orbit, including the perigee and the apogee.

Since the mass doesn’t change, when the radius is small, the velocity is large, and vice versa. Thus, the velocity at the perigee is always greater than at the apogee.

21. How much weight does each pyramid support when holding up this beam weighing 600 kg?

(A) \( A = 200 \text{ kg}, B = 400 \text{ kg} \)  
(B) \( A = 220 \text{ kg}, B = 380 \text{ kg} \)  
(C) \( A = 240 \text{ kg}, B = 360 \text{ kg} \)  
(D) \( A = 260 \text{ kg}, B = 340 \text{ kg} \)  
(E) \( A = 280 \text{ kg}, B = 320 \text{ kg} \)

The center of gravity for the beam is the middle of the beam, 5 meters from either end. The downward force exerted at the center of gravity is 600 kg times the acceleration of gravity, or 600 N.

Because the beam is not moving up or down, the downward force of 600 N must be equal to an upward force exerted by A and B together.

If A and B together exert a combined upward force of 600 N, then whatever A’s force is, it must be 600 N minus the force exerted by B.

\[ A + B = 600 \text{ N} \quad A = 600 \text{ N} - B \]

A and B do not exert an equal upward force, because as you know from sitting on a see-saw, A can exert less force than B because A is further from the beam’s center of the gravity. The upward force by A is perpendicular to the radius of rotation. A perpendicular force times the radius of rotation is called the “moment of force.”
Because the beam is stationary, and not rotating around its midpoint, the moment of force exerted by A equals the moment of force exerted at B. 3 meters times the upward force at A equals 2 meters times the upward force at B.

We now have two equations. One equation is that the 3 meters times the upward force at A equals 2 meters times the upward force at B.

The other equation is that the upward force at A equals 600 N minus the upward force at B.

Substituting the value of A into three times A, B turns out to be exerting an upward force of 360 kg, and A is therefore exerting an upward force of 240 kg.

\[ 3A = 2B \]
\[ 3(600 \text{ kg} - B) = 2B \]
\[ B = 360 \text{ kg} \quad A = 240 \text{ kg} \]

22. If the center of gravity for a triangle is where the three median lines intersect, and a median line is the line that bisects an angle, how high up the vertical height of a triangle is the center of gravity?

The center of gravity for a triangle is where the three median lines intersect. The median line bisects an angle.

Three triangles are created by the three median lines, and each triangle equals one-third of the area of the large triangle.

The area of a triangle is one-half base times height.

The area of small triangle is \( \frac{1}{2} \) base times little \( h \) (\( \frac{1}{2} Bh \)).

The area of this small triangle is one-third the area of the large triangle, \( \frac{1}{2} \) base times the height, capital \( H \), of the large triangle \( \frac{1}{3} (\frac{1}{2} BH) \).

Canceling out one-half and \( B \), little \( h \) equals one-third of capital \( H \).

\[ h = \frac{1}{3} H \]

Thus, the center of gravity of a triangle is where the three medians intersect, which is one-third of the triangle’s height.
Test, Lesson 5 – Circular Motion- Answer Key

23. Where is the center of gravity for this retaining wall?

(A) 1.4m to the right of the vertical side and 2.4m up from the base
(B) 1.6m to the right of the vertical and 2.6m up from the base
(C) 1.8m to the right of the vertical and 2.4m up from the base
(D) 1.6m to the right of the vertical and 2.4m up from the base
(E) 1.8m to the right of the vertical and 2.6m up from the base

The center of gravity of an object is the point where, when suspended from the center of gravity, the object does not rotate, meaning no net torque is being exerted by the center of gravity on the axis of rotation. The reason no torque is being exerted is that the center of gravity is at the exact same spot as the axis of rotation.

When figuring out the center of gravity, any point can be proposed as the axis of rotation. We can then calculate the specific X and Y distance of the center of gravity from the proposed axis of rotation. For a retaining wall, we’ll propose the axis of rotation to be point A at the inside edge of the base.

The distance of the retaining wall’s center of gravity to the proposed axis of rotation can be divided into its X and Y components.

We first need to divide the cross section of the retaining wall into a rectangle and a triangle. The center of gravity for the rectangle is the exact center of the rectangle, 3 meters up and half a meter in from the side.

The center of gravity for the triangle is one-third of the way up and one-third of the way in from the side, which is 2 meters up from its base and 1 meter to the right of its height.

The plan now is to calculate the distance of each center of gravity to the axis of rotation in the X and Y directions. In the X direction, the distance of the rectangle’s center of gravity to the axis of rotation is 0.5 meters. The distance of the triangle’s center of gravity to the axis of rotation is 2 meters.
In the Y direction, the distance of the rectangle’s center of gravity to the axis of rotation is 3 meters. The distance of the triangle’s center of gravity to its axis of rotation is 2 meters.

We will now calculate the torque exerted by the rectangle and triangle in the X direction. The torque exerted by the rectangle is its area times its distance to the axis of rotation. When the torque for the rectangle and triangle in the X direction are added together, they equal the torque exerted by the entire retaining wall in the X direction. We repeat this for the Y direction to obtain the X and Y distances from the center of gravity to the axis of rotation for the entire retaining wall.

\[ A_{\text{rectangle}} \times 0.5 + A_{\text{triangle}} \times 2 = A_{\text{retaining wall}} \times Y \]

The cross-sectional area of the retaining wall is 15 square meters – 6 square meters for the rectangle and 9 square meters for the triangle.

Here we go. The cross-sectional area of the rectangle, 6 meters square, times its distance to the axis of rotation in the X direction, 0.5 meters, plus the area of the triangle, 9 meters square, times its distance in the X direction, 2 meters, equals the area of the retaining wall, 15 meters squared, times its unknown distance to the axis of rotation.

The center of gravity for the retaining wall works out to be 1.4 meters up from its left side.

When we do the same thing in the Y direction, the center of gravity for the retaining wall works out to be 2.4 meters up from its base.

24. Where should the fulcrum be placed to balance the 30 pound weight and the 10 pound weight hanging from the ends of this 6 foot long rod?

(A) 1.0 foot to the right of the 30 pound weight
(B) 1.5 feet to the right of the 30 pound weight
(C) 2.0 feet to the right of the 30 pound weight
(D) 2.5 feet to the right of the 30 pound weight
(E) 3.0 feet to the right of the 30 pound weight

When the rod is balanced, force times the distance to the left of the fulcrum equals force times the distance to the right of the fulcrum.
If we call the distance of the 30 pound weight to the fulcrum “X,” then the distance of the 10 pound weight to the fulcrum has to be 6 minus X, because the rod is only 6 feet long.

30 pounds times X has to equal 10 pounds time six minus X.

\[30X = 10(6 - X)\]
\[40X = 60\]
\[X = 1.5 \text{ ft.}\]
\[6 - X = 4.5 \text{ ft.}\]

X works out to be 1.5 feet, and 6 minus X, 4 and a half feet.

25. Here is a man lifting a set of barbells. The barbell to his left weighs 60 kg and the one to his right weighs 40 kg. His right arm, being stronger, lifts with a force of 85 kg while his left arm can only lift with a force of 65 kg. The distance between the two barbells is 1.8 meters. His hands are 0.6 meters apart. Where along the barbell should he place his hands so that the barbells remain balanced?

The first step is to find the center of gravity for the barbells. Whatever spot we choose, the distance to one side will be \(C_G\), and the other side, 1.8 meters minus \(C_G\).

To be balanced, the force times the distance on each side of the center of gravity must be the same. So 40 N times the distance to the center of gravity, represented by \(C_G\), must equal 60 N times 1.8 meters minus \(C_G\). The distance to the center of gravity works out to be 1.08 meters from the smaller barbell and 0.72 meters from the larger one.
If we now find the center of gravity for the two hands, we can place that center of gravity over the barbell’s center of gravity, and we’ll have our answer because we know that the two hands are 0.6 meters apart.

Force times distance for the right hand is 85 times the distance to the center of gravity, represented by X. Force times distance for the left hand is 65 times 0.6 minus X. X turns out to be 0.26 meters from the right hand, and therefore 0.34 meters from the left hand.

\[85X = 65(0.6 - X)\]
\[150X = 39\]
\[X = 0.26\text{m}\]

That means the right hand should be 1.08 meters minus 0.26, or 0.82 meters from the right barbell and his left hand 0.72 minus 0.34, or 0.38 meters from the left barbell.

\[0.6\text{m} - 0.26\text{m} = 0.34\]